Orientation-Preserving Vectorized Distance Between Curves

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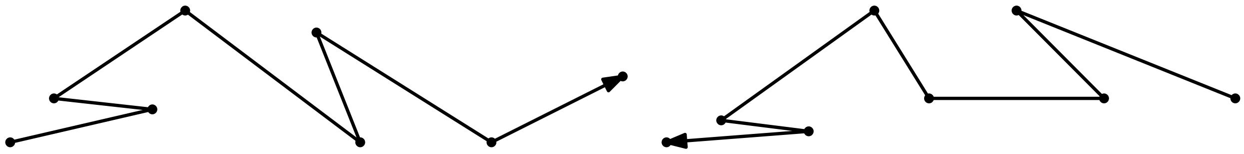
Distances on Trajectories and Their Efficiency

Consider two trajectories with m waypoints.



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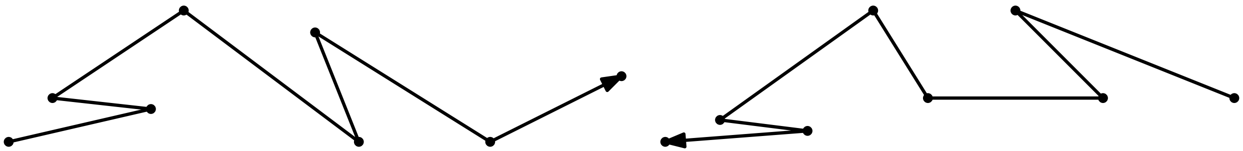

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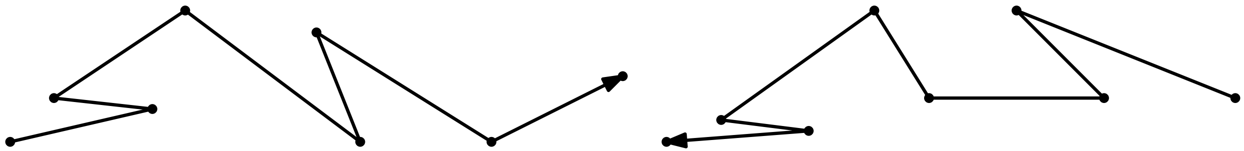

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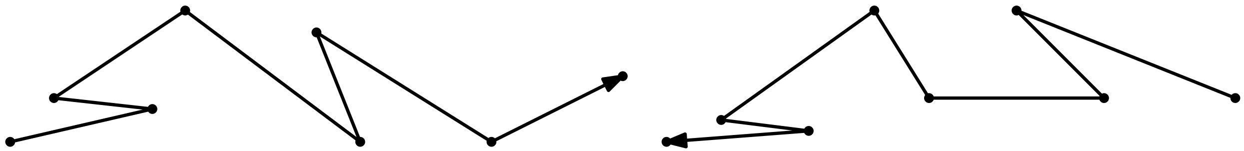

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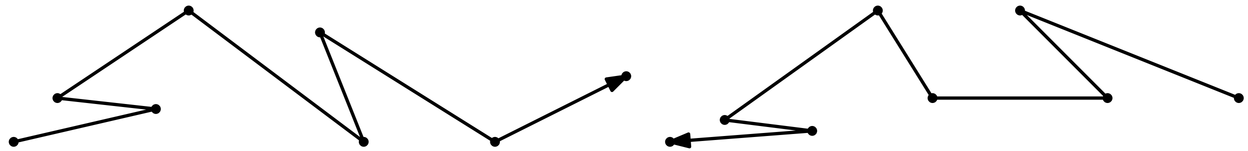

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  All above 3 preserve orientation; Complexity $\sim O(m^2)$.

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Properties of a distance we are interested in:

- It does not depend on m.
- It is as fast as calculating dot products in Euclidean spaces.
- It provides an embedding for curves in a Euclidean space in order to enable the use of ML algorithms.

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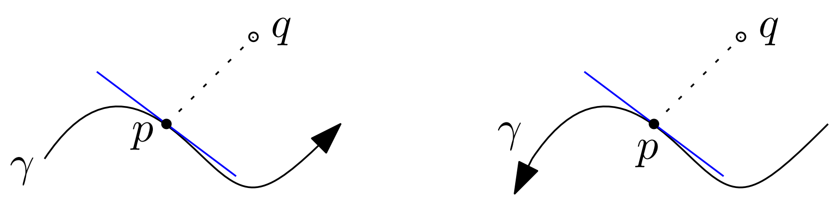

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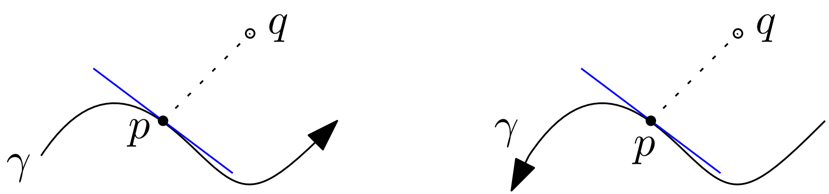

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For $Q = \{q_1, \ldots, q_n\} \subset \mathbb{R}^2$, we get $v_Q^{\mathrm{mD}} : \{\mathrm{curves}\} \to \mathbb{R}^n$ by

$$v_Q^{\mathrm{mD}}(\gamma) = (v_{q_1}^{\mathrm{mD}}(\gamma), \cdots, v_{q_n}^{\mathrm{mD}}(\gamma)).$$

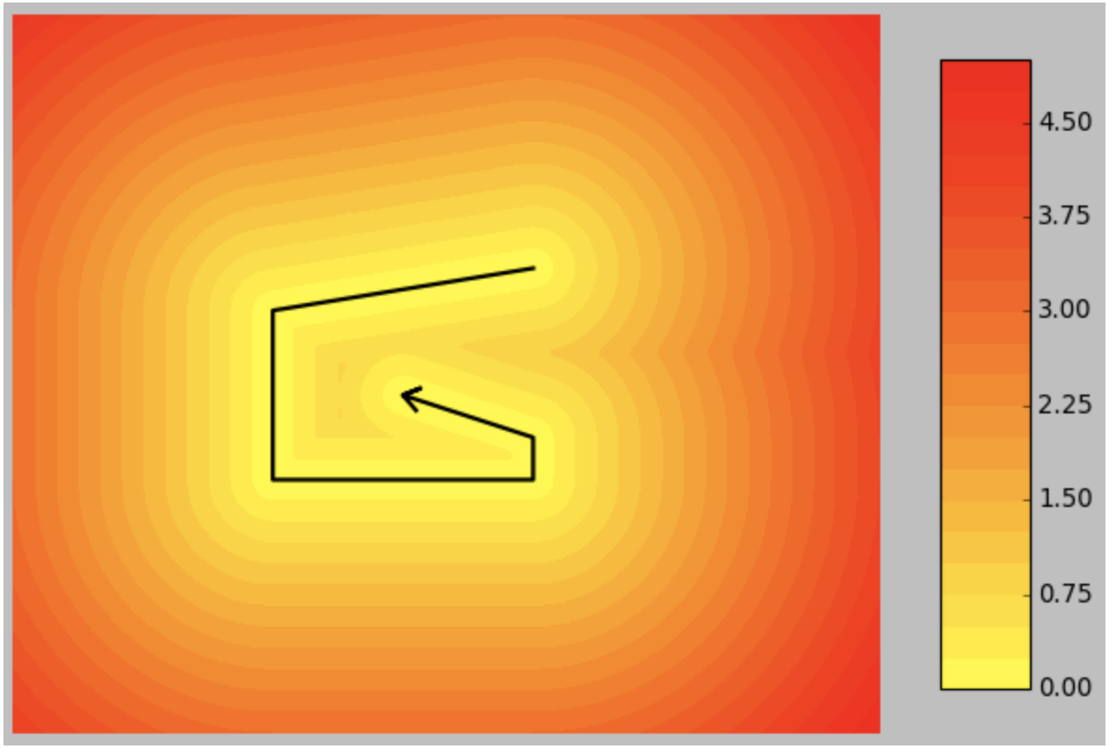

Figure: An example of MinDist function v_q^{mD} for a curve.

MinDist Distance

Let γ, γ' be two curves and $Q = \{q_1, \ldots, q_n\} \subset \mathbb{R}^2$. Then

$$\mathrm{d}_Q^{\mathrm{mD}}(\gamma, \gamma') = \frac{1}{\sqrt{n}} \| v_Q^{\mathrm{mD}}(\gamma) - v_Q^{\mathrm{mD}}(\gamma') \|.$$

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Question

How can we encode orientation preserving property into $\mathrm{d}_Q^{\mathrm{mD}}$?

Definitions and Notation

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The subset of Γ containing all simple curves.

- n_p

Considering the direction of curve, n_p is the unique normal at p.

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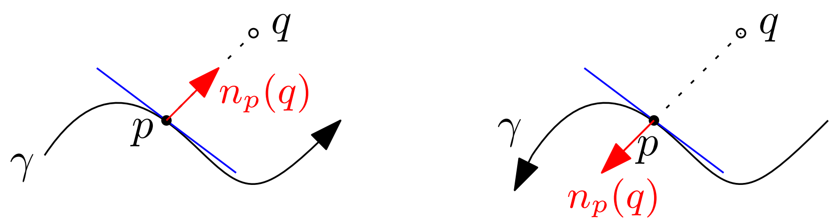

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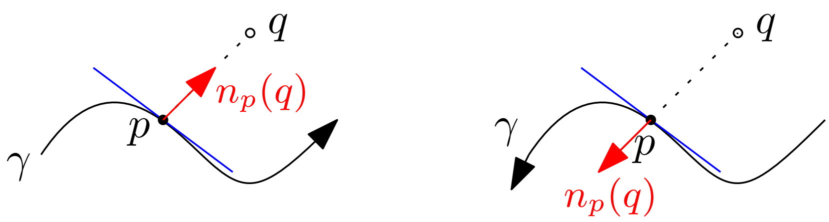


For endpoints we set

$$v_q^\sigma(\gamma) = \frac{1}{\sigma} \langle n_p, q - p \rangle \frac{\|q\|_{\infty,p}}{\|q-p\|} e^{-\frac{\|q-p\|^2}{\sigma^2}},$$

where $\|q\|_{\infty,p}$ is the l^∞-norm of q in the coordinate system with axis parallel to n_p and L (tangent line at p) and origin at p.

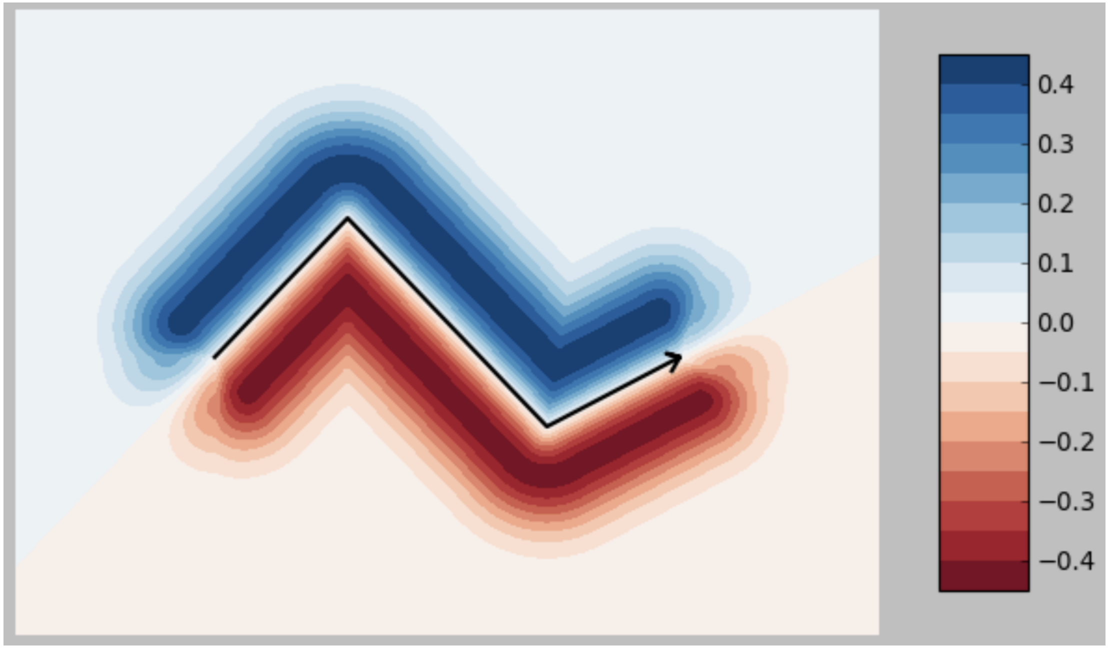
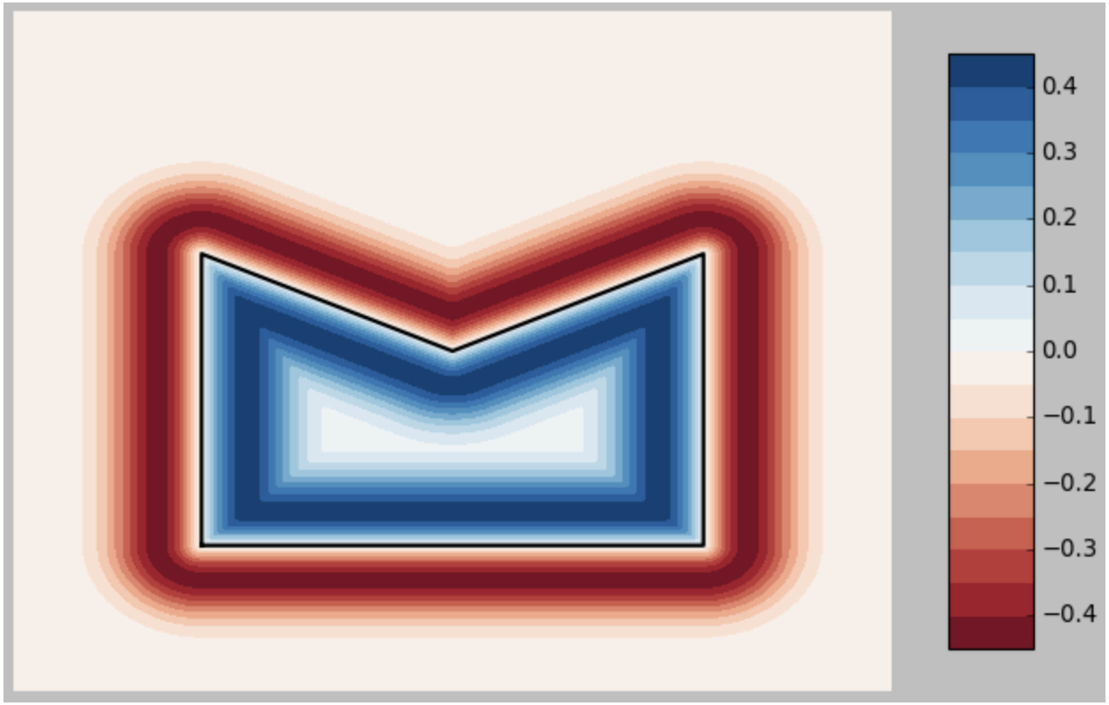


Figure: An example of v_q^σ function for a curve with sidedness encoded by positive/negative values.



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This embedding enables using ML algorithms, which is the biggest advantage of our work.

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In Practice

In practice, however, we found that $|Q| = 20$ is usually enough.

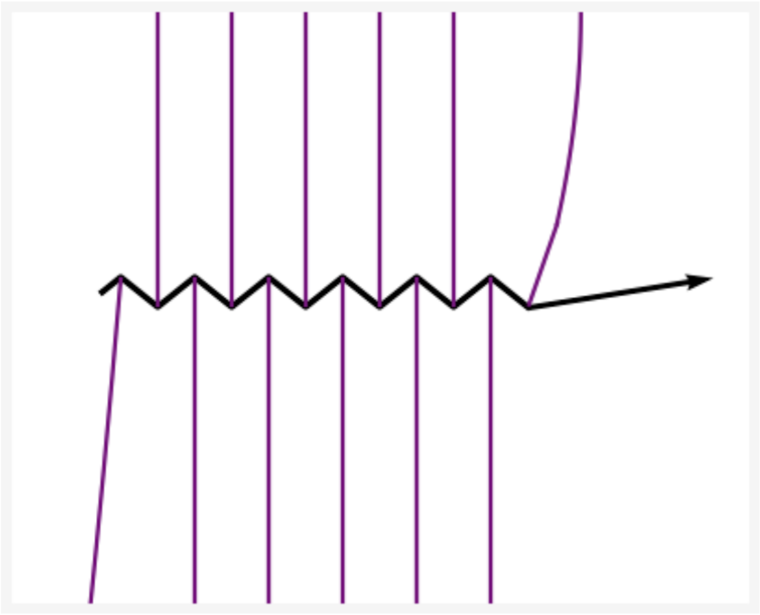
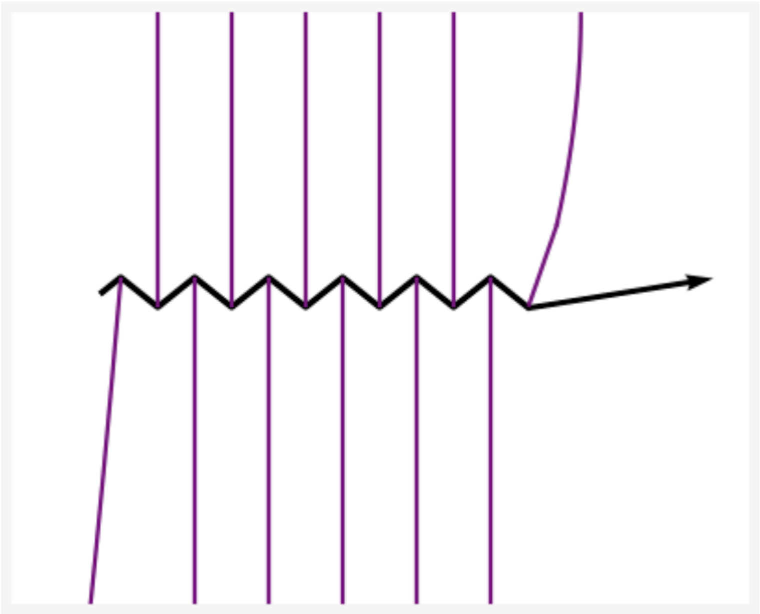


Figure: Medial axis in purple



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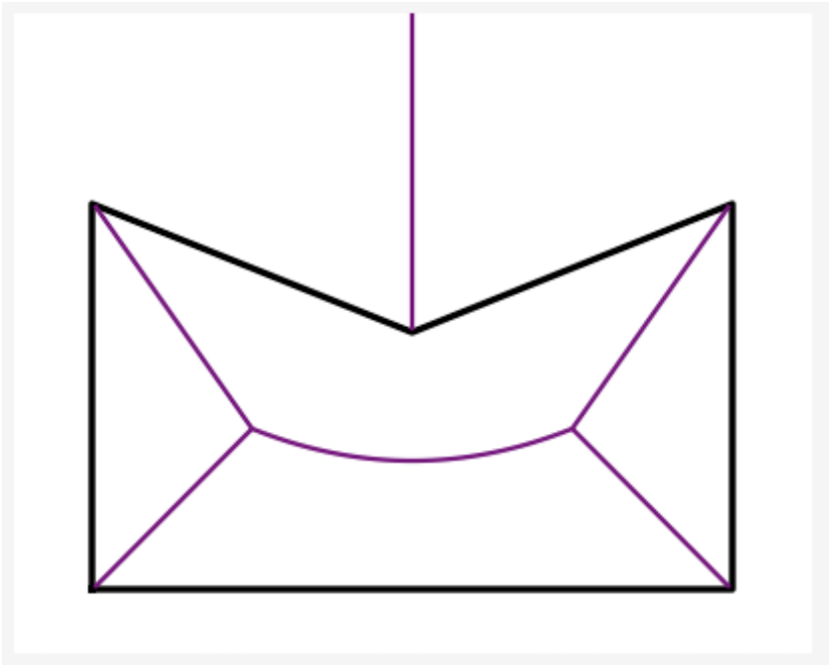
As we can see, the MA captures the noise in a curve.



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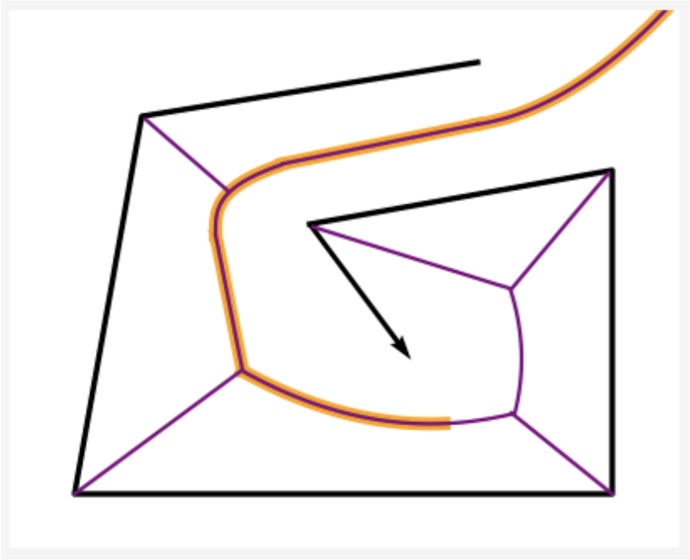

Figure: Medial axis in purple and Signed medial axis in orange

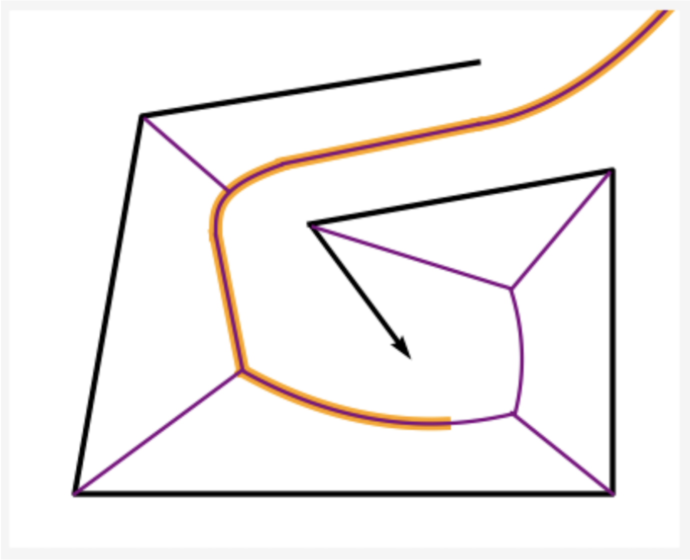

Figure: Medial axis in purple and Signed medial axis in orange

The notion of SMA comes to play to capture the sidedness of curve, i.e. capturing equidistance points with different v_q^σ signs.

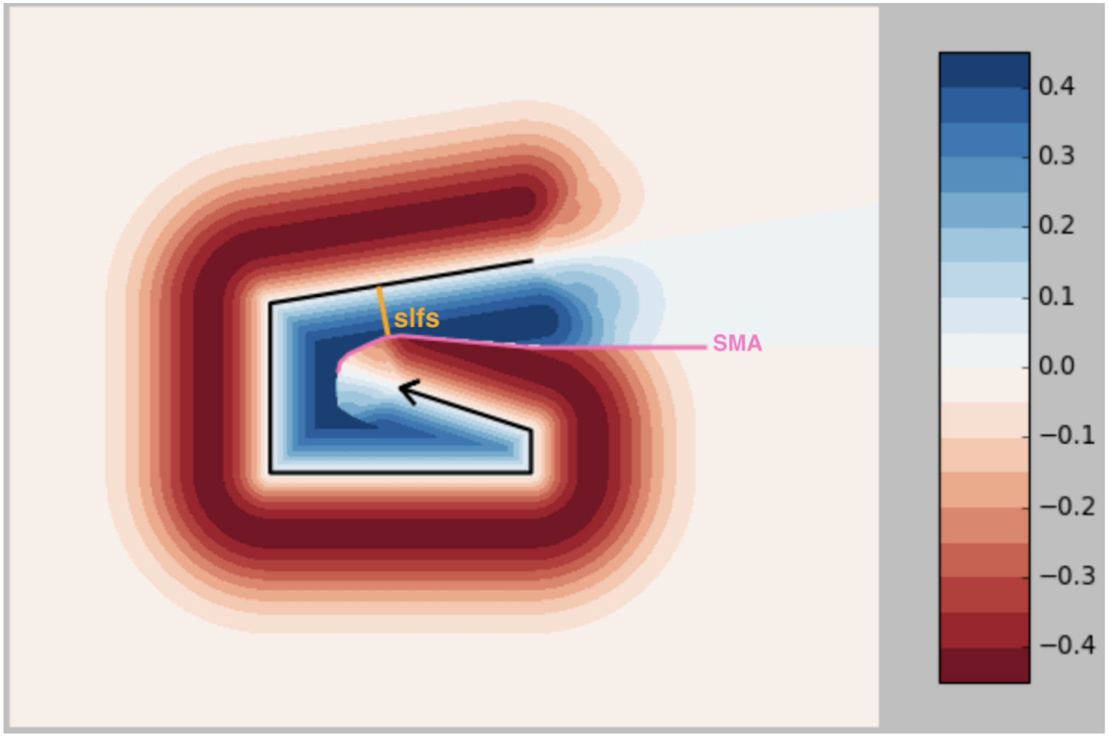

Figure: Signed medial axis in pink and Signed local feature size in orange

Landmark stability

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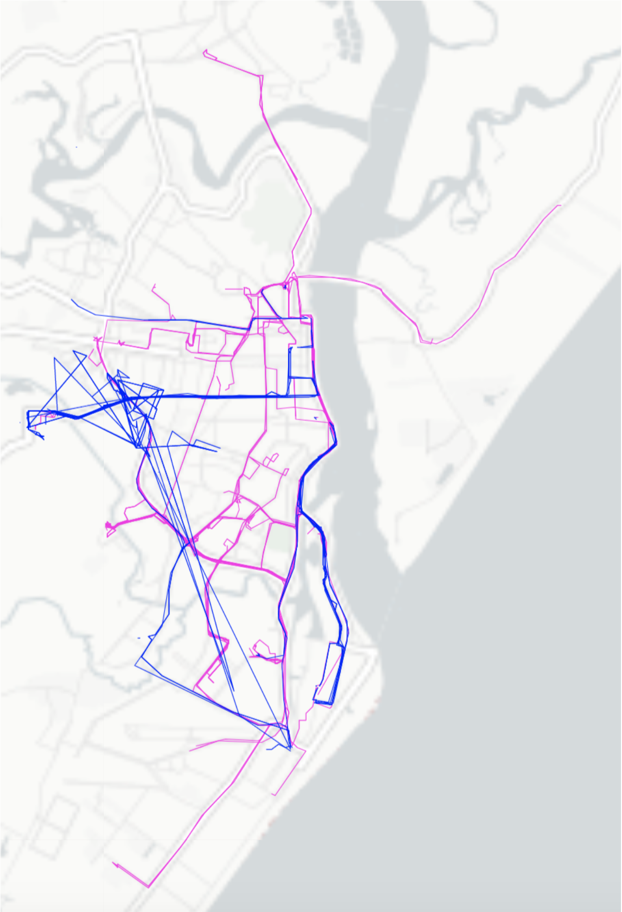
Theorem

$$\mathtt{d}_Q^{\mathrm{mD}}(\gamma, \gamma') \leq \mathtt{d}_H(\gamma, \gamma').$$

Average test errors with v_Q^{σ} and v_Q^{mD} vectorizations

	Feature Mapping	v_Q^{σ}	v_Q^{mD}
	Classifier	**Test Error**	**Test Error**
Car-Bus	Linear SVM	0.361	0.361
	Gaussian SVM	0.225	0.302
	Decision Tree	0.230	0.212
	Random Forest	**0.157**	0.183
Characters	Linear SVM	0.018	0.040
	Gaussian SVM	0.012	0.038
	Decision Tree	0.018	0.074
	Random Forest	**0.010**	0.049
Pigeons	Linear SVM	**0.003**	0.506
	Gaussian SVM	0.006	0.517
	Decision Tree	0.016	0.516
	Random Forest	0.006	0.524

Buses: blue, Cars: pink



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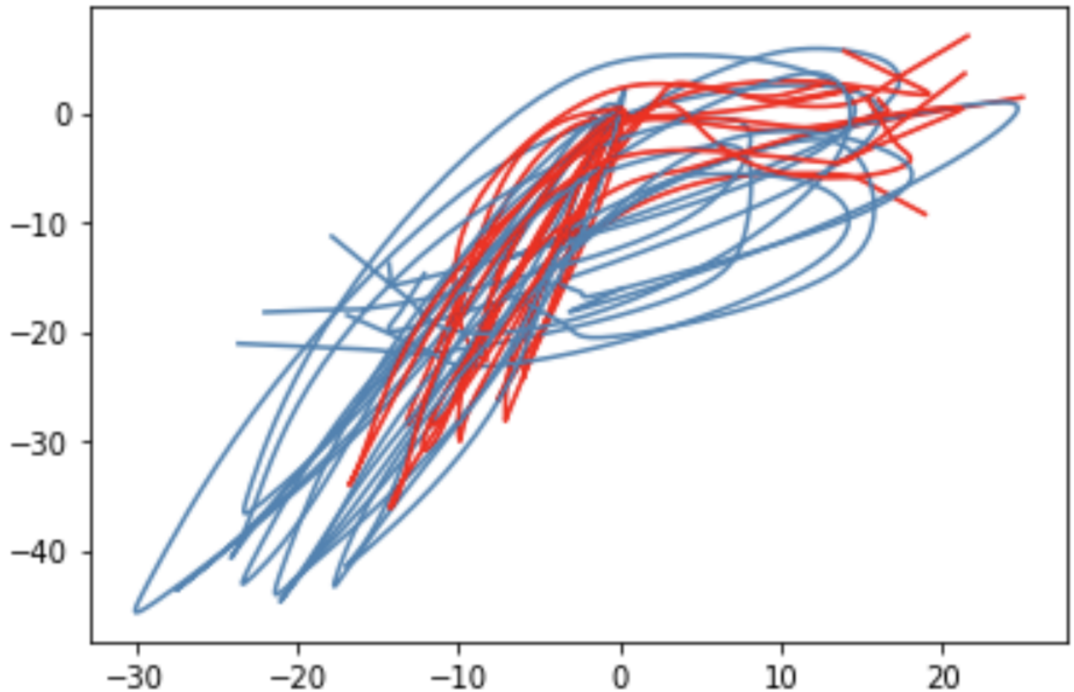
Letters p and r



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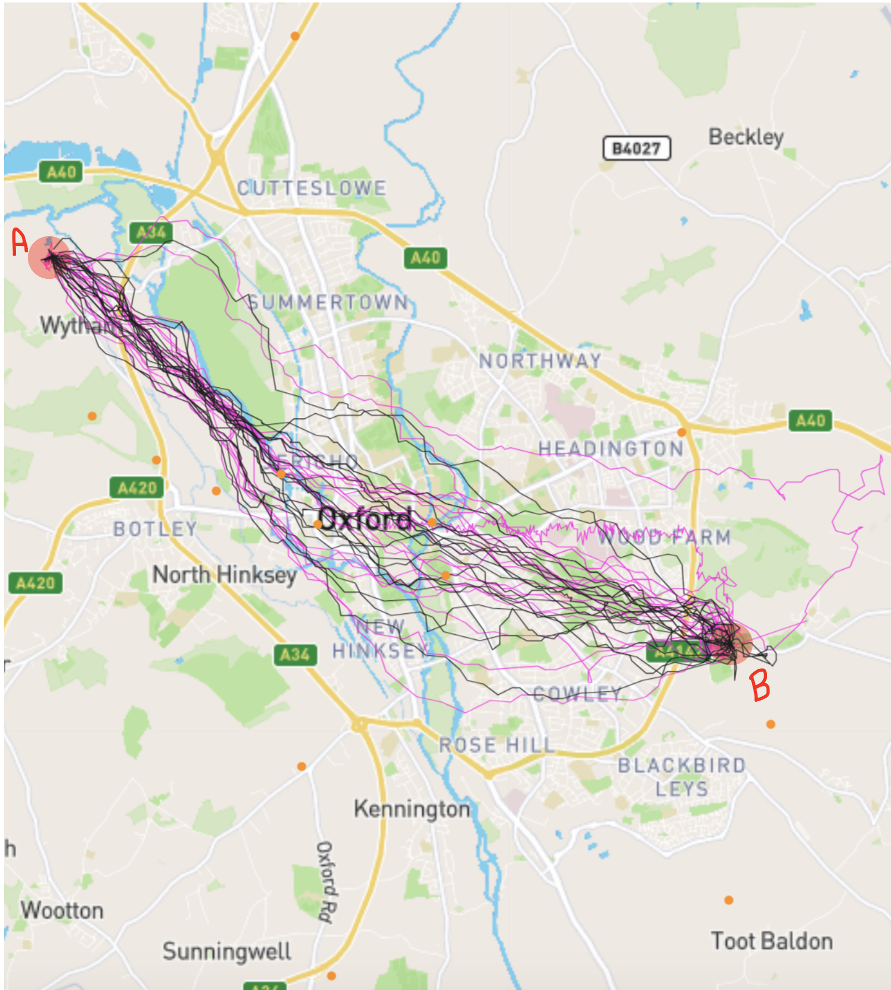
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NORTHWAY
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HEADINGTON
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Oxford
WOOD FARM
A420
North Hinksey
NEW HINKSEY
A34
COWLEY
ROSE HILL
BLACKBIRD LEYS
Kennington
Oxford Rd
Wootton
Sunningwell
A34
Toot Baldon

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Pigeons	Linear SVM	**0.003**	0.506
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Using the implemented codes

The related package is available in Python package index (PyPI) via the package

trjtrypy

Just: pip install trjtrypy

Jeff M. Phillips and Hasan Pourmahmood-Aghababa, Orientation-Preserving Vectorized Distance Between Curves, Mathematical and Scientific Machine Learning (MSML), August 2021.

Jeff M. Phillips and Pingfan Tang, Simple Distances for Trajectories via Landmarks, ACM GIS SIGSPATIAL, 2019.

Jeff M. Phillips and Pingfan Tang, Sketched MinDist, International Symposium on Computational Geometry, 2020.

Thank You!